

Scalar and fermion contributions to the vacuum energy

Dimitrios Metaxas

*Department of Physics,
National Technical University of Athens,
Zografou Campus, 15780 Athens, Greece
metaxas@central.ntua.gr*

Abstract

I consider a theory of a real scalar and a fermion field, with a Yukawa interaction and a potential term that admits two degenerate minima at the tree level. I calculate the quantum vacuum energy difference between these two vacua and I find a finite, non-zero result, with scalar and fermion contributions whose origin and physical significance I discuss.

I will start by reviewing the problem of the vacuum energy for renormalizable quantum field theories, in four-dimensional flat spacetime, that contain a scalar field, ϕ , which is endowed, at tree level, with a standard kinetic term and a general potential term, $U(\phi)$, which is bounded below.

(A) If the potential term at hand has a single minimum (vacuum) at $\phi = \phi_{\min}$ then quantization can be performed around it after expanding $U(\phi) = U(\phi_{\min}) + \frac{1}{2}U''(\phi_{\min})(\phi - \phi_{\min})^2 + \dots$, discarding the constant term, using the quadratic term to describe a scalar excitation of mass m around the minimum, with $m^2 = U''(\phi_{\min})$, and treating the higher order terms in perturbation theory as interactions with the respective coupling constants.

The constant term, also called the vacuum energy term, along with the mass and the coefficients of the higher order interactions have no meaning at this point; they are called bare terms and get regularized by (infinite) multiplications or subtractions, along with a similar treatment of the kinetic term in the usual process of renormalization.

Associated with this procedure are two parameters, both with dimensions of mass: Λ , which is used in order to cut-off divergent expressions, and μ , that sets the scale in which the physical parameters of the theory, masses and coupling constants are defined or measured. Then one proceeds by calculating, order by order in the perturbation expansion, the various Green's functions of the theory as well as the related functional expressions of the effective action with the corresponding effective potential [1].

The cut-off, Λ , was just a mathematical convention and should be absent from any final result of these calculations. The theory is defined by specifying the values of the masses and the coupling constants at a scale μ ; although the Green's functions and the effective action depend on the scale, any physical result derived from them should be μ -independent. For example: one may measure and define the masses and coupling constants of the theory using scattering experiments at a “reference” scale $\mu_{ref} = 1\text{GeV}$. Then one may predict and measure the outcomes of experiments at any other scale, say, $\mu_{exp} = 10\text{GeV}$. The result should be the same with what one would have obtained after having used a different μ_{ref} to start with. This is embodied in the renormalization group formalism and is expressed mathematically by the fact that the total derivative of any physical quantity with respect to μ , given by the sum of the various partial derivatives, must vanish.

We see immediately the reason why the constant, vacuum energy, term was discarded: there is no physical process or experiment that depends on

it; it can be set to zero, or any other value if one is not worried about the semiclassical expansion around an infinite constant. Once this is done (here I will consider it set to zero) there is no prediction for a different value, nor can there be any process to verify such a prediction. If one wants to use the renormalization group formalism consistently, however, one must take care of the constant term too, that is, in our case, subtract its value at the minimum at any level in the perturbation expansion of the effective potential [2].

When the theory under consideration is coupled to gravity, whether the latter is considered at the classical level or quantized, the value of the vacuum energy becomes a physical observable that can be measured in the cosmological expansion rate and contributes to the cosmological constant [3]. The quantum theory of gravity is not renormalizable; it can be viewed as an effective quantum field theory [4], with a limited range of predictability, as all effective quantum field theories, and its implications will not be considered here. As far as renormalizable quantum field theories are concerned, there can be no prediction for the vacuum energy defined as the value of the renormalized effective potential at its minimum.

It is sometimes argued that the sum of the zero-point energies of the field modes at the minimum contributes

$$\frac{1}{4\pi^2} \int_0^\Lambda dk k^2 \sqrt{k^2 + m^2} \approx \frac{\Lambda^4}{16\pi^2} \quad (1)$$

when a momentum cut-off regularization scheme is employed, or

$$\frac{\mu^{4-d}}{(2\pi)^{(d-1)}} \frac{1}{2} \int d^{d-1}k \sqrt{k^2 + m^2} \approx \frac{m^4}{64\pi^2} \ln \left(\frac{m^2}{\mu^2} \right) \quad (2)$$

when dimensional regularization and minimal subtraction prescription are performed. In (2), a fermion field would have given a contribution with the opposite sign, involving, of course, the fermion mass at the minimum. The cut-off, Λ , is usually considered to be related to the Planck or a Grand Unified Theory (GUT) scale, and the scale μ to the radiation associated with the supernova observations or the Cosmic Microwave Background [5].

Although these expressions are suggestive of contributions to the vacuum energy that drive it away from a zero value when non-renormalizable interactions such as gravity are considered, they can hardly be considered as a prediction of a renormalizable quantum field theory. Higher energy scales,

such as the GUT scale, may or may not leave an imprint on processes at the electroweak scale depending on the details of the decoupling procedure, none of the contributions, however, may depend explicitly on the cut-off in a way implied by (1). As far as the expression in (2) is concerned, one also sees that it cannot, by itself, correspond to a well-defined prediction; it is rather a one-loop result that should be subtracted if the perturbation expansion around the vacuum is to be done consistently.

(B) Let us now consider the case where the potential energy term, $U(\phi)$, has, besides the global minimum at ϕ_{\min} , a second, local minimum at ϕ_{met} , such that $U(\phi_{\text{met}}) > U(\phi_{\min})$. This local minimum corresponds to a “false”, metastable vacuum, and the energy difference between the two vacua is a physical observable that can, in principle, be measured if an appropriate metastable state is prepared. The perturbation expansion of the effective potential must account for this fact; the renormalization group equation [2] will ensure that the vacuum energy difference can be consistently defined, the value of the “true” vacuum energy, however, is still undetermined and can be set to zero. Only the energy difference between the two vacua is a meaningful, physical quantity.

This vacuum energy difference is also an input of the theory, much like the various masses and coupling constants; it is not a prediction of the quantum field theory. Similar considerations apply when the global minimum of the potential was not present at tree level but was induced by radiative, quantum effects [1]. The dimensionful parameter that defines the location of the absolute minimum and its related energy difference with respect to the metastable one is again an input of the theory, although “camouflaged” at the tree level. As an additional, important note for these cases, one should mention that the energy of a metastable state has an imaginary part that is related to the rate of its decay [6]; this is a non-perturbative effect, however, and will not show up at any level of the perturbation expansion.

(C) One may also consider a theory where the potential energy term at tree level has a discrete or continuous family of degenerate minima that are related by a symmetry. Two simple examples that one can have in mind involve a complex scalar field with a “Mexican-hat” potential, or a real scalar field with a “reflection” symmetry of $\phi \rightarrow -\phi$. Quantization can again be performed picking one of these minima and following the same procedure as above. The value of the potential at the minimum is again undefined and can be consistently set to zero. Once this is done, by symmetry considerations,

the value of the renormalized potential at any other minimum will be zero as well.

(D) Finally, coming to the case that is relevant to the present work, one can imagine the case of a potential term with a set of degenerate minima that have the same value of the energy at tree level but are not otherwise related by any symmetry. A simple example would be a potential with two minima at ϕ_1 and ϕ_2 , such that $U(\phi_1) = U(\phi_2)$ but $U''(\phi_1) \neq U''(\phi_2)$. Then the elementary excitations around each minimum would have different masses. If one were to pick one minimum, say ϕ_1 , to quantize the theory, all the subtractions described before would have to be performed at this point, and the difference of terms such as (2) around the two minima should give a finite, possibly non-zero result for ϕ_2 . This would be a definite prediction for the energy of the second vacuum, similar to well-known phenomena like the Casimir effect [7]. Obviously, it is not possible to have a renormalizable quantum field theory in four dimensions with such a potential term at tree level (it is interesting, however, that the effective potential in the Standard Model allows for the possibility of a second minimum, other than the one in the electroweak scale, close to the Planck scale and degenerate in energy [8]). Even so, there are other examples where asymmetries between classically degenerate vacua can be seen and this is investigated further below.

In order to examine a simple case of the aforementioned asymmetries, I will consider here a theory with a real scalar and a fermion field with a Yukawa interaction and the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - U(\phi) + i\bar{\psi}\not{\partial}\psi - g\phi\bar{\psi}\psi, \quad (3)$$

where the potential term,

$$U(\phi) = \frac{\lambda}{4!}\phi^2(\phi - \phi_0)^2, \quad (4)$$

has two degenerate minima at $\phi = 0$ and $\phi = \phi_0$.

There are two sources of asymmetry in this case: first, as it is obvious, the fermion acquires a mass, $m_f = g\phi_0$, around the second minimum, while it is massless around the first. Second, the scalar potential is, in fact, asymmetric in field space. The masses of the scalar excitations are the same around the two vacua,

$$U''(0) = U''(\phi_0) = \frac{\lambda}{12}\phi_0^2 \equiv M^2, \quad (5)$$

since renormalization involves a scale, μ , however, there is a resulting asymmetry between the zero and the non-zero vacuum, depending on where the renormalization conditions are imposed. As a final result, we will find, therefore, a difference in the renormalized vacuum energies of these two vacua, although they are degenerate at tree level.

The effective potential at one loop, after dimensional regularization, is given by the well-known expression

$$U_{\text{eff}}(\phi) = U(\phi) + \frac{1}{64\pi^2} \left[(U'')^2 \left(\ln \frac{U''}{\mu^2} - \frac{1}{2} \right) - 4g^4\phi^4 \left(\ln \frac{g^2\phi^2}{\mu^2} - \frac{1}{2} \right) \right] \quad (6)$$

$$+ c_0 + c_1\phi + c_2\frac{\phi^2}{2} + c_3\frac{\phi^3}{3!} + c_4\frac{\phi^4}{4!}.$$

I have included the four counterterms, proportional to c_4, c_3, c_2, c_1 , in order to impose the four renormalization conditions

$$U_{\text{eff}}''''(\phi_0) = \lambda, \quad (7)$$

$$U_{\text{eff}}'''(\phi_0) = \frac{\lambda\phi_0}{2}, \quad (8)$$

$$U_{\text{eff}}''(\phi_0) = M^2, \quad (9)$$

and

$$U_{\text{eff}}'(\phi_0) = 0, \quad (10)$$

and the constant, c_0 counterterm, to account for the vacuum energy. One is only allowed a single counterterm to adjust that, and once a condition is imposed at one vacuum there is a definite, possibly non-zero prediction, for the value at the other vacuum.

All the other counterterms, from linear to quartic are allowed since there is no symmetry, like reflection with respect to the origin (evenness of the potential), which is usually imposed for simplicity. The linear counterterm is not necessary since it corresponds merely to a shift of the field, it has been included, however, for clarity. Using it, I have imposed the conditions that keep ϕ_0 as one of the minima. Then the second minimum will be slightly displaced from $\phi = 0$. This effect can also be calculated from the linear term in the potential, for small enough values of the couplings, however, as will be seen shortly, this will be a subleading effect.

I should mention at this point that I consider values of the couplings that do not destroy the vacuum structure of the theory, that is I take the Yukawa coupling small enough, $g^2 < \lambda/4$, as is required for stability.

The four renormalization conditions stated above can be solved to give the four coefficients, c_4, c_3, c_2 and c_1 , and the final result for the effective potential at one loop, without including the c_0 term, is:

$$\begin{aligned}
U_{\text{eff}}(\phi) = U(\phi) &+ \frac{1}{64\pi^2} \left[(U'')^2 \left(\ln \frac{U''}{\mu^2} - \frac{1}{2} \right) - 4g^4\phi^4 \left(\ln \frac{g^2\phi^2}{\phi_0^2} - \frac{1}{2} \right) \right] \\
&+ \frac{1}{64\pi^2} \left[\left(\frac{1}{12}\lambda^2\phi_0^3 \ln \frac{M^2}{\mu^2} + \frac{3}{2}\lambda^2\phi_0^3 - \frac{32}{3}g^4\phi_0^3 \right) \phi \right. \\
&- \left(\frac{1}{3}\lambda^2\phi_0^2 \ln \frac{M^2}{\mu^2} + \frac{13}{4}\lambda^2\phi_0^2 - 24g^4\phi_0^2 \right) \phi^2 \\
&+ \left(\frac{1}{2}\lambda^2\phi_0 \ln \frac{M^2}{\mu^2} + 3\lambda^2\phi_0 - 32g^4\phi_0 \right) \phi^3 \\
&- \left. \left(\frac{1}{4}\lambda^2 \ln \frac{M^2}{\mu^2} + \lambda^2 - \frac{44}{3}g^4 \right) \phi^4 \right]. \quad (11)
\end{aligned}$$

Now we have a definite expression for the value of the potential at ϕ_0 :

$$U_{\text{eff}}(\phi_0) = \frac{1}{64\pi^2} \left[M^4 \left(\ln \frac{M^2}{\mu^2} - \frac{1}{2} \right) + \frac{1}{4}\lambda^2\phi_0^4 - 2g^4\phi_0^4 \right]. \quad (12)$$

The second minimum, as mentioned before, is not located exactly at $\phi = 0$, its position, however, can be calculated in the small coupling expansion, and the effect of its displacement on the vacuum energy can be seen to be subleading compared to

$$U_{\text{eff}}(0) = \frac{1}{64\pi^2} M^4 \left(\ln \frac{M^2}{\mu^2} - \frac{1}{2} \right). \quad (13)$$

The displacement of the second minimum from zero can be shown to be of order $\lambda\phi_0$, and the resulting change in the vacuum energy of order $\lambda^3\phi_0^4$.

One has, therefore, a definite prediction for the vacuum energy difference between the two vacua,

$$\delta U = U_{\text{eff}}(\phi_0) - U_{\text{eff}}(0) = \frac{1}{64\pi^2} \left(\frac{1}{4}\lambda^2\phi_0^4 - 2g^4\phi_0^4 \right), \quad (14)$$

regardless of the choice of c_0 (which can be chosen so as to cancel the term of (13) for consistency [2]). This is a quantum result that was absent at tree level, where one would have to put in by hand the value of the vacuum energy, or even any vacuum energy difference between two or more vacua.

Before embarking on the discussion of the result, I should mention that, as is well known, there is a region in field space where the final expression for the one-loop effective potential has an imaginary part [9]. It is the region where $U''(\phi) < 0$, and one has to be more careful when deriving physical results associated with this part of the field space. Our areas of interest, however, near $\phi = 0$ and $\phi = \phi_0$, have no overlap with the problematic region in this case.

Now we can proceed to investigate the origins and implications of the final result. As far as the second, fermion contribution to (14) is concerned, one might have expected the result qualitatively, as well as its sign. It is also interesting, however, that one has a non-zero scalar contribution to the vacuum energy difference. This arises from the fact that the potential is asymmetric with respect to the renormalization conditions imposed. This is true even without the fermion field; a fermion without a mass term at tree level was considered here merely in order to get a simple and suggestive quantitative result.

Without the fermion one can equally well impose the previous renormalization conditions at $\phi = 0$; then the second minimum near a non-zero ϕ_0 would show the same effect, that is an energy difference equal to the first factor in (14). This calculation is easy to do and will not be reproduced here. The final result for just the scalar field with the potential term in (4) is that the vacuum at ϕ_0 in the quantum theory has higher energy than the one at 0 by the amount given by the first term in (14), regardless of where the renormalization conditions are imposed.

With the fermion term used here, and the condition $g^2 < \lambda/4$ that has to be fulfilled for stability, one sees that the energy at ϕ_0 is always higher than that at $\phi = 0$. It should be kept in mind, however, that the model considered is quite simple and that even slightly more elaborate models, with more fermion species or fermion mass terms will give a more general expression with a greater range of final values.

It is important that the final result in (14) is independent of any cut-off or renormalization scale, and is given, as expected, by the parameters that define the theory, couplings and masses. Any “running” from renormalization

group effects appears at higher orders as it should. One may also view the expression derived here as a definite finite result that comes from considering finite factors included in (2) and then taking the difference of two such terms. In any case, it is a prediction of a renormalizable quantum field theory and a purely quantum effect.

The fact that the two, classically degenerate, vacua are energetically inequivalent because of quantum corrections, gives this simple model a structure that is richer than expected. The vacuum with higher energy, ϕ_0 in this case, becomes metastable, although it was classically stable. One can accordingly calculate its rate of decay; the appropriate formalism is related to the results of [10] although the physical situation here is different. Since the vacuum energy difference is a quantum effect, the result for this vacuum decay rate is extremely small; it is proportional to the exponential of minus the “bounce” action, which, in our case, turns out to be of order $1/\lambda^2$. It would be interesting, as a problem for further research, to study the evolution of the vacua and the effective potential in a finite temperature and cosmological setting in this or related problems where the breaking or the lack of symmetry play an important role [11].

As a final note I should discuss the possibility of a “landscape” of vacua, a large number of which are degenerate, with zero energy at the classical level or even after some quantum corrections have been taken into account. Unless they are all related by the same symmetries, it does not seem possible to have zero energy in all of them when higher order quantum effects are considered, and the energy difference between two adjacent vacua, if one literally translates the results obtained here, would be proportional to powers of coupling constants times their distance in field space. It is an attractive scenario which states that if the value of the vacuum energy of a particular minimum is fixed by some reason to be zero, the value of the vacuum energy for any nearby minimum will be a highly suppressed and calculable number.

One frequently encounters the problem, however, that some of the interactions that are involved, in this or other physically important situations, are non-renormalizable, the most important example being the gravitational interaction; when these are regarded as effective quantum field theories [4], instead of a coupling constant expansion that was the basic tool of renormalizable theories, one now has an expansion in powers of the energy, and it is possible that well-defined results for the vacuum energy or energy difference exist in these situations as well. It would be interesting, therefore, as

a subject of future work, to consider the results of similar considerations in effective quantum field theories .

Acknowledgements

This work was completed while visiting the National Technical University of Athens. I would like to thank the people of the Physics Department for their hospitality.

References

- [1] S. Coleman, *Aspects of Symmetry*, Cambridge Univ. Press (1985).
S. Coleman and E. J. Weinberg, *Phys. Rev.*, **D7**, 1888 (1973).
E. J. Weinberg, hep-th/0507214.
- [2] C. Ford, D. R. T. Jones, P. W. Stephenson and M. B. Einhorn, *Nucl. Phys.*, **B395**, 17 (1993).
M. B. Einhorn and D. R. T. Jones, *JHEP*, **0704**, 051 (2007).
- [3] S. Weinberg, *Rev.Mod.Phys.*, **61**, 1 (1989).
T. Padmanabhan, *Phys. Rep.*, **380**, 235 (2003).
- [4] J. F. Donoghue, *Phys. Rev.*, **D50**, 3874 (1994).
M. M. Anber, J. F. Donoghue and M. El-Houssieny, *Phys.Rev.*, **D83**, 124003 (2011).
J. F. Donoghue, arXiv:1209.3511 [gr-qc].
- [5] J. Martin, *Comptes Rendus Physique*, **13**, 566 (2012).
I. L. Shapiro and J. Sola, arXiv:0808.0315 [hep-th].
- [6] S. Coleman, *Phys. Rev.*, **D15**, 2929 (1977).
A. D. Linde, *Nucl. Phys.*, **B216**, 421 (1983).

- [7] G. Plunien, B. Muller and W. Greiner, *Phys.Rept.*, **134**, 87 (1986).
K. A. Milton, S. A. Fulling, P. Parashar, A. Romeo, K.V. Shajesh and J. A. Wagner, *J.Phys.*, **A41**, 164052 (2008).
- [8] M. Sher, *Phys.Rept.*, **179**, 273 (1989).
C. D. Froggatt and H. B. Nielsen, *Phys.Lett.*, **B368**, 96 (1996).
D. L. Bennett and H. B. Nielsen, *Int.J.Mod.Phys.*, **A9**, 5155 (1994).
J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori, A. Riotto and A. Strumia, *Phys.Lett.*, **B709**, 222 (2012).
F. Bezrukov, M. Yu. Kalmykov, B. A. Kniehl and M. Shaposhnikov, *JHEP*, **1210**, 140 (2012).
I. Masina, arXiv:1209.0393 [hep-ph].
F. Bezrukov, G. K. Karananas, J. Rubio and M. Shaposhnikov, arXiv:1212.4148 [hep-ph].
R. Armillis, A. Monin and M. Shaposhnikov, arXiv:1302.5619 [hep-th].
- [9] R. Jackiw, *Phys.Rev.*, **D9**, 1686 (1974).
E. J. Weinberg and A. Wu, *Phys. Rev.*, **D36**, 2474 (1987).
- [10] E. J. Weinberg, *Phys. Rev.*, **D47**, 4614 (1993).
D. Metaxas and E. J. Weinberg, *Phys. Rev.*, **D53**, 836 (1996).
- [11] D. Metaxas, *Phys. Rev.*, **D63**, 083507 (2001).
D. Metaxas, *Phys. Rev.*, **D75**, 047701 (2007).
J. Alexandre, *Int.J.Mod.Phys.*, **A26**, 4523 (2011).
J. Alexandre and A. Tsapalis, *Phys.Rev.*, **D87** 025028 (2013).
K. Farakos, *Int.J.Mod.Phys.*, **A27**, 1250168 (2012).
K. Farakos and D. Metaxas, *Phys.Lett.*, **B711**, 76 (2012).